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# The Manin constant of an optimal quotient of $J_0(431)$

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**Abstract**

We exhibit an example of an optimal quotient of $J_0$ (431) for which the Manin constant is Z.



*Keywords:* Manin constant; Modular forms; Modular abelian varieties

## 1. Introduction

In this paper, we exhibit an example of an optimal quotient of $J_0(N)^{\mathrm{new}}$ for which the Manin constant is not 1. This provides a counterexample to a conjecture made by Agashe in his thesis [2, 2.2.8] and suggests the correct conjecture is that stated in [3].

Let $N$ be a positive integer and denote by $J_0(N)_{/\mathbf{Q}}$ the jacobian of $X_0(N)$, the modular curve over $\mathbf{Q}$ which classifies isomorphism classes of elliptic curves over $\mathbf{Q}$ together with a given cyclic subgroup of order $N$. The Hecke algebra $\mathbf{T}_N$ is defined to be the subring of the ring of endomorphisms of the jacobian $J_0(N)$ generated by the Hecke operators $T_p$ for $p \nmid N$ and $U_p$ for $p \mid N$. Following the terminology of [9, 2(d)], we call an abelian variety $A$ over $\mathbf{Q}$ an *optimal quotient of* $J_0(N)$ if there is a surjective map $J_0(N) \longrightarrow A$ (over $\mathbf{Q}$) which has connected kernel. We define an *optimal quotient of* $J_0(N)^{\mathrm{new}}$ similarly.

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Let $I$ be an ideal in $\mathbf{T}_N$ such that $\mathbf{T}_N/I$ is torsion-free. The quotient $A(I) := J_0(N)/IJ_0(N)$ is an abelian variety defined over $\mathbf{Q}$, which is an optimal quotient of $J_0(N)$. In particular, if $f$ is a weight 2 newform for $\Gamma_0(N)$ and $I_f$ is the annihilator of $f$ in the Hecke algebra, then the quotient $A(I_f)$ is called the *Shimura quotient associated to* $f$. It has dimension $g = [K_f : \mathbf{Q}]$ where $K_f$ is the number field of least degree containing the coefficients of the $q$-expansion of $f$.

In [3, 2.2] Stein and Agashe define the Manin constant for optimal quotients of $J_0(N)$ as follows. Let $A = A(I)$ be as above and suppose it is of dimension $g$. Let $A_{/\mathbf{Z}}$ be the Néron model of $A$ over $\mathbf{Z}$. Then we have the diagram

$$\begin{array}{ccccc} \mathrm{H}^0(A_{/\mathbf{Q}}, \Omega^1_{A/\mathbf{Q}}) & \cong & \mathrm{H}^0(J_0(N)_{/\mathbf{Q}}, \Omega^1_{J_0(N)/\mathbf{Q}})[I] & \cong & S_2(\Gamma_0(N), \mathbf{Q})[I] \\ \cup & & & & \cup \\ \mathrm{H}^0(A_{/\mathbf{Z}}, \Omega^1_{A/\mathbf{Z}}) & & & & S_2(\Gamma_0(N), \mathbf{Z})[I] \end{array}$$

where $S_2(\Gamma_0(N), R)[I]$ denotes the set of weight 2 cusp forms for $\Gamma_0(N)$, whose $q$-expansions (a $+\infty$) have coefficients in the ring $R$, which are killed by $I$. So regarding $S_2(\Gamma_0(N), \mathbf{Z})[I]$ as contained in $\mathrm{H}^0(A_{/\mathbf{Q}}, \Omega^1_{A/\mathbf{Q}})$, there are two obvious lattices in $\mathrm{H}^0(A_{/\mathbf{Q}}, \Omega^1_{A/\mathbf{Q}})$ and taking wedge products gives

$$\bigwedge^g \mathrm{H}^0(A_{/\mathbf{Z}}, \Omega^1_{A/\mathbf{Z}}) = c \bigwedge^g S_2(\Gamma_0(N), \mathbf{Z})[I]$$

for some $c \in \mathbf{Q}^*$. Then the *Manin constant* $c_A$ of $A$ is defined to be the absolute value of $c$.

For the Shimura quotient associated to a weight 2 newform $f$ for $\Gamma_0(N)$ whose $q$-expansion coefficients lie in $\mathbf{Q}$, the dimension of $E = A(I_f)$ is 1. In this case $c_E$ is the classical Manin constant for elliptic curves, as originally introduced in [8]. In his paper, Manin also conjectures that $c_E = 1$ for any such quotient. In the direction of this conjecture, Edixhoven [4, Proposition 2] proves that $c_E$ is an integer, Mazur [9, Corollary 4.1] shows that if $p$ is a prime such that $p \mid c_E$, then $p^2 \mid 4N$ and in [1, Theorem A], Abbes and Ullmo show that if $p \mid c_E$, then $p \mid N$. It has been verified by Cremona that the Manin constant is 1 for at least every elliptic curve of conductor up to 1000.

In [3, Proposition 2.8] Agashe and Stein extend the result of Edixhoven to show that there is an injection

$$\mathrm{H}^0(A_{/\mathbf{Z}}, \Omega^1_{A/\mathbf{Z}}) \hookrightarrow S_2(\Gamma_0(N), \mathbf{Z})[I], \tag{1}$$

where $A = A(I)$ is an optimal quotient as above. Therefore the Manin constant $c_A$ of $A$ is just the order of the cokernel of this inclusion. That is to say,

$$c_A = [S_2(\Gamma_0(N), \mathbf{Z})[I] : \mathrm{H}^0(A_{/\mathbf{Z}}, \Omega^1_{A/\mathbf{Z}})].$$

In his thesis [12, Theorem 3.50], Stein generalises Mazur's theorem to optimal quotients of $J_0(N)^{\mathrm{new}}$ of arbitrary dimension. Making use of calculations involving the Jacobians of genus two curves in [5], Stein was also able to verify that the Manin constant is 1 for 28 certain 2-dimensional optimal quotients of $J_0(N)^{\mathrm{new}}$ (including some of non-square-free levels). Based on this, Agashe [2, 2.2.7] asks whether Abbes and Ullmo's above result generalises to arbitrary optimal quotients of $J_0(N)^{\mathrm{new}}$. He goes on to make the following conjecture.

**Conjecture 1** (*Agashe [2, 2.2.8]*). *For an optimal quotient of $J_0(N)^{\mathrm{new}}$, the Manin constant is* 1.

By using the example of prime level $N = 431$, which is of interest in the work of Kilford [7] on the failure of mod 2 multiplicity one, we will produce an optimal quotient of $J_0(431)$ for which the Manin constant is 2. This means that the answer to Agashe's question is no and we have provided a counterexample to Conjecture 1. Since we have a definition of a Manin constant for optimal quotients of arbitrary dimension, it is possible to consider products of such quotients. In order to produce our counterexample we consider the product, $E_1 \times E_2$, of two 1-dimensional Shimura quotients of $J_0(431)$. It is not in general true that the product of two optimal quotients is again optimal, but this turns out to be the case for our particular choice of elliptic curves. This leads us to believe that the following conjecture, stated in [3] is the correct one.

**Conjecture 2.** *For a Shimura quotient of $J_0(N)$ the Manin constant is* 1.

**Remarks.** (i) Comments made by Edixhoven to the authors of [3] suggest that for prime level $N$, the map $\mathrm{H}^0(J_0(N)_{/\mathbf{Z}}, \Omega^1_{J_0(N)/\mathbf{Z}}) \hookrightarrow S_2(\Gamma_0(N), \mathbf{Z})$ is surjective, and hence the Manin constant of $J_0(N)$ is 1. So for the case of $N = 431$, we have a sequence of subabelian varieties

$$E_1 \hookrightarrow E_1 \times E_2 \hookrightarrow J_0(431)$$

for which $c_{E_1} = 1$, $c_{E_1 \times E_2} = 2$ and $c_{J_0(431)} = 1$.

(ii) We have taken $N = 431$ here, but this is merely the first level at which mod 2 multiplicity one fails. Kilford [7] found further examples, at 503 and 2089. Using the same method, one can find 2-dimensional optimal quotients of $J_0(503)$ and $J_0(2089)$ for which the Manin constant is 2.

In [10], Stevens argues that when considering parametrisations of elliptic curves, $J_0(N)$ is not the correct parametrising object and should be replaced by $J_1(N)$ instead. He considers the Manin–Stevens constant of 1-dimensional quotients of $J_1(N)$ and makes the conjecture that $c_E = 1$ for any such quotient $E$, including non-optimal ones. Since $J_1(N)$ is a "better'' parametrising object, it may be wondered whether the example we have come up with is a phenomenon of $J_0(N)$ and the problem is solved by passing to $J_1(N)$, but this is not the case. Although the natural map $J_0(431) \longrightarrow J_1(431)$ is not an injection, our abelian variety $E_1 \times E_2$ still injects in to $J_1(431)$. To see this, we observe that the Shimura subgroup is cyclic and our chosen elliptic curves are isolated in their isogeny classes. Therefore $J_1(431) \longrightarrow E_1 \times E_2$ is an optimal quotient and our result is still valid; the Manin-Stevens constant of $E_1 \times E_2$ is 2. For a more detailed study of the Manin-Stevens constant of optimal quotients, see the author's forthcoming thesis.

## 2. Calculation

Take the level $N = 431$ and consider the jacobian $J = J_0(431)$. This has dimension 36 and is isogenous to a product of six Shimura quotients of dimensions 1,1,3,3,4 and 24. In Stein's tables [13] these quotients are labelled $431A1, \ldots, 431F1$, respectively. We shall consider the two elliptic curves $431A1$ and $431B1$, which we will denote $E_1$ and $E_2$. These curves have Weierstrass equations given by:

$$E_1 : y^2 + xy = x^3 - 1,$$

$$E_2 : y^2 + xy + y = x^3 - x^2 - 9x - 8.$$

We have maps $J \to E_1$ and $J \to E_2$ with connected kernels $I_{f_1}J$ and $I_{f_2}J$, respectively, where $f_1, f_2$ are newforms with rational coefficients. By dualising the above maps, we can regard $E_1$ and $E_2$ as subabelian varieties of $J$, and we have

$$0 \longrightarrow E_1 \cap E_1 \longrightarrow E_1 \times E_2 \cong (E_1 \times E_2)^\vee \longrightarrow J^\vee \cong J,$$

where for $B$ an abelian variety, $B^\vee$ denotes the dual abelian variety. A calculation with modular symbols shows that the curves $E_1$ and $E_2$ have trivial intersection inside the jacobian; see the table in [11]. So we see that the right-hand map is an injection, which in turn, implies the kernel of the natural map $J \longrightarrow E_1 \times E_2$ is connected. Therefore we can regard $E_1 \times E_2$ as an optimal quotient; in fact we have

$$E_1 \times E_2 = J/(I_{f_1} \cap I_{f_2})J.$$

This description allows us to compare the Manin constant of the product to the Manin constants of the factors $E_1$ and $E_2$. Let $\mathcal{E}_1$ (resp. $\mathcal{E}_2$) be the Néron model of $E_1$ (resp. $E_2$) over $\mathbf{Z}$. Then $\mathcal{E}_1 \times \mathcal{E}_2$ is the Néron model of $E_1 \times E_2$ over $\mathbf{Z}$. We have the inclusion

$$\mathrm{H}^0(\mathcal{E}_1 \times \mathcal{E}_2, \Omega^1_{(\mathcal{E}_1 \times \mathcal{E}_2)/\mathbf{Z}}) \hookrightarrow S_2(\Gamma_0(N), \mathbf{Z})[I_{f_1} \cap I_{f_2}]$$

as in (1) and wish to calculate the cokernel.

Since $\Omega^1_{(\mathcal{E}_1\times\mathcal{E}_2)/\mathbf{Z}} = p_1^*\Omega^1_{\mathcal{E}_1/\mathbf{Z}} \oplus p_2^*\Omega^1_{\mathcal{E}_2/\mathbf{Z}}$, where $p_1$ and $p_2$ are the projections on to the respective factors [6, II, Exercise 8.3], and cohomology commutes with flat base change [6, III, Proposition 9.3], we can decompose the zeroth cohomology group as

$$\begin{aligned}\mathrm{H}^0(\mathcal{E}_1\times\mathcal{E}_2, \Omega^1_{(\mathcal{E}_1\times\mathcal{E}_2)/\mathbf{Z}}) &= \mathrm{H}^0(\mathcal{E}_1\times\mathcal{E}_2, p_1^*\Omega^1_{\mathcal{E}_1/\mathbf{Z}}) \oplus \mathrm{H}^0(\mathcal{E}_1\times\mathcal{E}_2, p_2^*\Omega^1_{\mathcal{E}_2/\mathbf{Z}})\\ &= \mathrm{H}^0(\mathcal{E}_1, \Omega^1_{\mathcal{E}_1/\mathbf{Z}}) \oplus \mathrm{H}^0(\mathcal{E}_2, \Omega^1_{\mathcal{E}_2/\mathbf{Z}}).\end{aligned}$$

Due to the results of Mazur [9, Corollary 4.1] and Abbes and Ullmo [1, Theorem A] mentioned above, $c_{E_i} = 1$, and so

$$\mathrm{H}^0(\mathcal{E}_i, \Omega^1_{\mathcal{E}_i/\mathbf{Z}}) = S_2(\Gamma_0(N), \mathbf{Z})[I_{f_i}]$$

for $i = 1, 2$. Thus the Manin constant of $E_1 \times E_2$ is the order of the cokernel of the injection

$$S_2(\Gamma_0(N), \mathbf{Z})[I_{f_1}] \oplus S_2(\Gamma_0(N), \mathbf{Z})[I_{f_2}] \hookrightarrow S_2(\Gamma_0(N), \mathbf{Z})[I_{f_1} \cap I_{f_2}].$$

But this cokernel is non-trivial. The newforms $f_1$ and $f_2$ corresponding to $E_1$ and $E_2$ are congruent mod 2, which can be seen by comparing the first 72 $q$-expansion coefficients of $f_1 - f_2$ (the Sturm bound given in [14]). We actually have that

$$f_1 - f_2 = -2q^3 + 4q^5 + 2q^6 + O(q^7).$$

So the cusp form $(f_1 - f_2)/2$ is contained in $S_2(\Gamma_0(N), \mathbf{Z})[I_{f_1} \cap I_{f_2}]$, but it is clearly not contained in the direct sum $S_2(\Gamma_0(N), \mathbf{Z})[I_{f_1}] \oplus S_2(\Gamma_0(N), \mathbf{Z})[I_{f_2}]$. We have therefore shown

**Proposition 3.** *The Manin constant of* $E_1 \times E_2$ *is* 2.

This of course provides a counterexample to Conjecture 1 [2, 2.2.8] and means the answer to Agashe's question [2, 2.2.7] is no.

## References

[1] A. Abbes, E. Ullmo, À propos de la conjecture de Manin pour les courbes elliptiques modulaires, Compositio Math. 103 (3) (1996) 269–286.
[2] A. Agashe, The Birch and Swinnerton–Dyer formula for modular abelian varieties of analytic rank zero, Ph.D. Thesis, University of California, Berkeley, 2000.
[3] A. Agashe, W. Stein, The Manin constant, congruence primes and the modular degree, preprint.
[4] B. Edixhoven, On the Manin Constants of Modular Elliptic Curves, Arithmetic Algebraic Geometry (Texel 1989), Birkhäuser Boston, MA, 1991, pp. 25–39.
[5] E.V. Flynn, F. Leprévost, E.F. Schaefer, W.A. Stein, M. Stoll, J.L. Wetherall, Empirical evidence for the Birch and Swinnerton–Dyer conjectures for modular Jacobians of genus 2 curves, 1999, submitted for publication.

[6] R. Hartshorne, Algebraic Geometry, Springer, Berlin, 1977.
[7] L. Kilford, Some non-Gorenstein Hecke algebras attached to spaces of modular forms, J. Number Theory 97 (2002) 157–164.
[8] J.I. Manin, Parabolic points and zeta functions of modular curves, Izv. Akad. Nauk SSSR Ser. Mat. 36 (1972) 19–66.
[9] B. Mazur, Rational isogenies of prime degree, Invent. Math. 44 (2) (1978) 129–162.
[10] G. Stevens, Stickelberger elements and modular parametrizations of elliptic curves, Invent. Math. 98 (1) (1989) 75–106.
[11] W.A. Stein, Invariants of modular abelian varieties, available online at http://modular.fas.harvard.edu/Tables/papers.html, 1999.
[12] W.A. Stein, Explicit approaches to modular abelian varieties, Ph.D. Thesis, University of California, Berkeley, 2000.
[13] W.A. Stein, The Modular Forms Explorer, Available online at http://modular.fas.harvard.edu/mfd/mfe/.
[14] J. Sturm, On the congruences of modular forms, Lecture Notes in Mathematics, vol. 1240, 1987, pp. 275–280.